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Engineering Notes

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Concerning Lyapunov-Based Guidance

R. T. Yanushevsky*
Technology Service Corporation,
Silver Spring, Maryland 20910

I. Introduction

IN Refs. 1 and 2 the new guidance law design approach based on the Lyapunov method was considered. The described procedure enables us to create laws that improve the effectiveness of the proportional navigation law. The new laws will augment the proportional navigation (PN) law, which is a part of these laws. It was shown that the guidance law $u(t)$ (commanded acceleration) obtained for the linear planar model of engagement consists of the PN term and up to three additional terms and has the following form:

$$u(t) = N v_{\text{cl}}\dot{\lambda}(t) + N_1\dot{\lambda}^3(t) - N_2\lambda(t)\ddot{r}(t) + N_3 a_T(t)$$

$$N > 2,\; N_1 > 0 \qquad (1)$$

$$N_2 \begin{matrix}\geq 1\\ \leq 1\end{matrix} \quad \text{if} \quad \text{sign}\,[\ddot{r}(t)\dot{\lambda}(t)\lambda(t)] \begin{matrix}\leq 0\\ \geq 0\end{matrix}$$

$$N_3 \begin{matrix}\leq 1\\ \geq 1\end{matrix} \quad \text{if} \quad \text{sign}\,[a_T(t)\dot{\lambda}(t)] \begin{matrix}\leq 0\\ \geq 0\end{matrix}$$

where $a_T(t)$ is the target acceleration; $\lambda(t)$ and $\dot{\lambda}(t)$ are the line of sight (LOS) and LOS derivative, respectively; $r(t)$ is a range; v_{cl} is the closing velocity; N is the effective navigation ratio (as in the PN law); and N_1–N_3 are, in the general case, time-varying parameters.

The term $N_3 a_T(t)$ is different from the corresponding term in the augmented proportional navigation (APN) law because the parameter N_3 is time varying. The term $N_2\lambda(t)\ddot{r}(t)$ also influences the terminal velocity of a missile. The comparative analysis of the new guidance law with the traditional PN law given in Ref. 2 showed that the new guidance law guarantees shorter homing time requirements and larger capture area.

In Refs. 1 and 2 the expressions slightly different from Eq. (1) were obtained for the nonlinear planar model. These expressions have mostly pure theoretical rather than practical significance because, as it will be shown next, the real three-dimensional case embeds the planar case. The nonlinear planar model can be used to describe only the directed "upward" coordinate of commanded acceleration in the three-dimensional engagement model presented with respect to azimuth and elevation angles (e.g., see Ref. 3). Unfortunately, there was an error in the expressions mentioned [see Eq. (19) of Ref. 2, where λ should be changed for $\sin\lambda$], and, as a result, the guidance law contained a $1/\cos\lambda$ factor. The approach described in Refs. 1 and 2 was used in Ref. 4, which was focused on eliminating the singularity stipulated by $1/\cos\lambda$. This Note shows that the three-dimensional Lyapunov-based guidance law embeds the Lyapunov-based guidance laws obtained for the planar case.[1,2] It is also shown that the Lyapunov function used in Refs. 1 and 2 does not stipulate the guidance law with singularities.

II. Guidance for the Nonlinear Planar Model

For the nonlinear planar model of engagement, $\sin[\lambda(t)] = y(t)/r(t)$, where $y(t)$ is the relative separation between the missile and target perpendicular to the reference axis, so that

$$\ddot{\lambda}(t)\cos[\lambda(t)] - \dot{\lambda}^2(t)\sin[\lambda(t)] = -a_1(t)\sin[\lambda(t)] - a_2(t)\cos[\lambda(t)]\dot{\lambda}(t) + b_1\ddot{y}(t) \qquad (2)$$

or

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_2^2\tan x_1 - a_1(t)\tan x_1 - a_2(t)x_2 - \frac{b(t)}{\cos x_1}u + \frac{b(t)}{\cos x_1}f \qquad (3)$$

where $u(t)$ is a commanded missile acceleration, $x_1 = \lambda(t)$, $x_2 = \dot{\lambda}(t)$, and

$$a_1(t) = \ddot{r}(t)/r(t) \qquad (4)$$

$$a_2(t) = 2\dot{r}(t)/r(t) \qquad (5)$$

$$b(t) = 1/r(t) \qquad (6)$$

$$\ddot{y}(t) = -u(t) + a_T(t) \qquad (7)$$

The derivative of the Lyapunov function $Q = \frac{1}{2}cx_2^2$ (c is a positive number) along any trajectory of Eq. (3) equals

$$\dot{Q} = cx_2\left\{x_2^2\tan x_1 - a_1(t)\tan x_1 - a_2(t)x_2 - \frac{[b(t)u - b(t)a_T]}{\cos x_1}\right\}$$

or

$$\dot{Q} = c\left\{x_2^3\tan x_1 - a_1(t)x_2\tan x_1 - a_2(t)x_2^2 - \frac{[b(t)x_2u - b(t)x_2a_T]}{\cos x_1}\right\} \qquad (8)$$

The negative definiteness of $\dot{Q}$ can be guaranteed by the control-guidance law

$$u = N v_{\text{cl}}\cos[\lambda(t)]\dot{\lambda}(t) + N_1\cos[\lambda(t)]\dot{\lambda}^3(t) - N_2\sin[\lambda(t)]\ddot{r}(t) - N_0 r(t)\dot{\lambda}^2(t)\sin[\lambda(t)] + N_3 a_T(t) \qquad (9)$$

$$N > 2, \qquad N_1 > 0, \qquad N_0 \begin{matrix}\geq 1\\ \leq 1\end{matrix} \quad \text{if} \quad \text{sign}\,[\dot{\lambda}(t)\lambda(t)] \begin{matrix}\leq 0\\ \geq 0\end{matrix}$$

$$N_2 \begin{matrix}\geq 1\\ \leq 1\end{matrix} \quad \text{if} \quad \text{sign}\,[\ddot{r}(t)\dot{\lambda}(t)\lambda(t)] \begin{matrix}\leq 0\\ \geq 0\end{matrix}$$

$$N_3 \begin{matrix}\leq 1\\ \geq 1\end{matrix} \quad \text{if} \quad \text{sign}\,[a_T(t)\dot{\lambda}(t)] \begin{matrix}\leq 0\\ \geq 0\end{matrix}$$

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*Senior Scientist.

As it follows from the preceding consideration, the guidance law has no singularities.

Comparing the guidance law of Eq. (9) with the nonlinear guidance of Ref. 4, we can conclude that Eq. (9) presents a wider class of nonlinear missile guidance than in Ref. 4. The components of the guidance law considered in Ref. 4 coincide with the corresponding ones in Eq. (9) when $N_i = 1$. Such a case can result in unrobustness of the missile guidance system. The time-varying gains in Eq. (3) guarantee its robustness and also increase the effectiveness of the proposed guidance law.

III. Three-Dimensional Engagement

For the three-dimensional case and the Earth-based coordinate system, the line of sight and its derivative can be represented as

$$\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}_1(t)\boldsymbol{i} + \lambda_2(t)\boldsymbol{j} + \lambda_3(t)\boldsymbol{k} \tag{10}$$

$$\dot{\boldsymbol{\lambda}}(t) = \dot{\boldsymbol{\lambda}}_1(t)\boldsymbol{i} + \dot{\lambda}_2(t)\boldsymbol{j} + \dot{\lambda}_3(t)\boldsymbol{k} \tag{11}$$

where $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ are unit vectors along to the north, up, and east coordinate axis, respectively,

$$\lambda_s(t) = R_s/r \qquad (s = 1, 2, 3) \tag{12}$$

$$\ddot{\lambda}_s(t) = -a_1(t)\lambda_s(t) - a_2(t)\dot{\lambda}_s(t) + b_1(t)[a_{\mathrm{Ts}}(t) - u_s] \tag{13}$$

R_s $(s = 1, 2, 3)$ are the RTM-vector coordinates (RTM means range r between a target and a missile), $a_{\mathrm{Ts}}(t)$ are the coordinates of the target acceleration vector, and $u_s(t)$ are the coordinates of the missile acceleration vector, which are considered as controls. [Equation (13) is obtained analogous to Refs. 1 and 2; see also a linear part of Eq. (3) assuming $\cos x_1 = 1$ and $tgx_1 = x_1$.]

The Lyapunov function is chosen as the sum of squares of the LOS derivative components that corresponds to the nature of proportional navigation.

$$Q = \frac{1}{2}\sum_{s=1}^{3} d_s\dot{\lambda}_s^2 \tag{14}$$

where d_s are positive coefficients.

Its derivative can be presented in the following form:

$$2\dot{Q} = \sum_{s=1}^{3} d_s\ddot{\lambda}_s\dot{\lambda}_s \tag{15}$$

or

$$2\dot{Q} = \sum_{s=1}^{3} d_s\left\{-a_1(t)\lambda_s\dot{\lambda}_s - a_2(t)\dot{\lambda}_s^2 + b_1(t)\dot{\lambda}_s[a_{\mathrm{Ts}}(t) - u_s]\right\} \tag{16}$$

Analogous to Ref. 2, the controls $u_s(t)$that guarantee $\lim\|\dot{\lambda}\| \to 0, t \to \infty$ can be presented as

$$u_s = Nv_{\mathrm{cl}}\dot{\lambda}_s + \sum_{k=1}^{3} u_{s_k} \tag{17}$$

where

$$u_{s1}(t) = N_1\dot{\lambda}_s^3(t) \qquad N_1 > 0 \tag{18}$$

$$u_{s2}(t) = N_{2s}\lambda_s(t)\ddot{r}(t) \tag{19}$$

$$N_{2s} \begin{matrix}\geq 1\\ \leq 1\end{matrix} \qquad \text{if} \qquad \operatorname{sign}[\ddot{r}(t)\dot{\lambda}_s(t)\lambda_s(t)] \begin{matrix}\leq 0\\ \geq 0\end{matrix} \tag{20}$$

$$u_{s3}(t) = N_{3s}a_{\mathrm{Ts}}(t) \tag{21}$$

$$N_{3s} \begin{matrix}\leq 1\\ \geq 1\end{matrix} \qquad \text{if} \qquad \operatorname{sign}[a_{\mathrm{Ts}}(t)\dot{\lambda}_s(t)] \begin{matrix}\leq 0\\ \geq 0\end{matrix} \qquad s = 1, 2, 3 \tag{22}$$

Equations (17–22) were obtained similar to the linear planar case [see Eq. (1)]. However, for the three-dimensional engagement model, in the case $d_s = 1$ the term

$$\sum_{s=1}^{3} -a_1(t)\lambda_s\dot{\lambda}_s$$

in Eq. (16) equals zero. This means that controls $u_{s2}(t) = N_{2s}\lambda_s(t)\ddot{r}(t)$ are not needed to guarantee $\lim\|\dot{\lambda}\| \to 0, t \to \infty$. Nevertheless, the just-mentioned controls are very important parts of the guidance law.

The commanded acceleration can be considered as consisting of two components—radial and tangential. As it follows from Eq. (19), the components $u_{s2}(t)$ belong to the radial acceleration, that is, they influence the closing velocity.

Usually, during a missile flight only two LOS rate components are dominant so that the case of equal d_s is not typical and then $u_{s2}(t)$ $(s = 1–3)$ also influence the tangential acceleration. However, the radial component is dominant.

Controls $u_{s2}(t) = N_{2s}\lambda_s(t)\ddot{r}(t)$ $(s = 1–3)$ do not influence the tangential component of a missile acceleration in the case of equal d_s; they change the radial acceleration component, which is important to guarantee an appropriate acceleration (force) at the moment of intercept.

For many types of existing missiles (e.g., without throttlable engines), radial acceleration cannot be utilized as a control action. Such missiles are not able to use thrust control as a part of a guidance law. Controls u_{s2} can influence a missile trajectory only by decelerating its motion.

As seen from Eqs. (17–22), the multidimensional PN law follows immediately as one of the possible solutions and as a component of a more complicated law with nonlinear terms. The PN law reacts almost identically on various changes of LOS rate (assuming that the closing velocity does not vary drastically), that is, small and fast changes of LOS result in proportional changes of acceleration. According to Eqs. (16–18), by increasing N in the PN law, we can more rapidly decrease the LOS rate. But this will increase the level of noise when the LOS rate becomes small, and, hence, the accuracy of guidance is decreased. Moreover, big gains can make the whole guidance system unrobust. From a pure physical consideration we can assume that the system with a variable gain, which is bigger when LOS rate is big and smaller when LOS rate is small, will act better than the traditional PN system. The component of Eq. (18) with a properly chosen N_1 serves this purpose.[1,2] The term of Eq. (21) distinguishes from the APN term because of the time-varying gain [Eq. (22)]. Properly chosen, each of the just-mentioned additional terms enables us to increase the performance of the guidance system.

The developed guidance laws can be used for the midcourse and terminal guidance. During the midcourse stage, the components of the LOS are obtained from Eq. (12). For the terminal stage these components are usually calculated based on measurements of azimuth and elevation angles. The vectors $\boldsymbol{\lambda}(t)$ and $\dot{\boldsymbol{\lambda}}(t)$ can be presented as (e.g., see Ref. 3)

$$\boldsymbol{\lambda} = \begin{bmatrix}\cos\alpha\cos\beta\\ \cos\alpha\sin\beta\\ \sin\alpha\end{bmatrix}$$

$$\dot{\boldsymbol{\lambda}} = \begin{bmatrix}-\sin\alpha\cos\beta\\ -\sin\alpha\sin\beta\\ \cos\alpha\end{bmatrix}\dot{\alpha} + \begin{bmatrix}-\cos\alpha\sin\beta\\ \cos\alpha\cos\beta\\ 0\end{bmatrix}\dot{\beta} \tag{23}$$

where α and β are elevation and azimuth angles.

By comparing the guidance law Eq. (17) using Eq. (23) with the guidance law for the nonlinear planar model Eq. (9), we can conclude that Eq. (9) can be used to analyze the coordinate u_3 in the three-dimensional case.

The developed guidance law was compared with the kappa midcourse guidance law.[5] The kappa algorithm is based on knowledge of the predicted intercept point. Because this is a terminal optimal problem, it requires complete information (current and future) about the missile and target as well as the external conditions during the engagement. However, the missile dynamic equations including drag and lift, etc. can only be implemented approximately, and it is impossible to estimate analytically the influence of incomplete information on the outcome of the engagement. The predicted intercept point and time-to-go are important parameters that dominate the accuracy of the optimal solution. These parameters can only be estimated, and it is difficult to evaluate the influence of the errors on the engagement results.

The new developed guidance law was compared with the kappa guidance law based on the existing flight-out tables for SM-2 missiles obtained for various target positions. The detailed simulation model of engagement including missile thrust and drag components, and the autopilot model, was used. The guidance law $u_s = 3v_{\mathrm{cl}}\dot{\lambda}_s - N_{2s}\lambda_s\ddot{r} + [1.25v_{\mathrm{cl}}\dot{\lambda}_s(t)]^3$, $N_{2s} = (1.5, 0.95)$ was chosen for the midcourse stage, whereas for the homing stage the proportional navigation law $u_s = 4v_{\mathrm{cl}}\dot{\lambda}_s$ was used. [The launch parameters were chosen so that the trajectories were in a plane, that is, $u_2(t) = 0$; $u_3(t)$ contained an additional term to compensate the influence of gravity.]

The simulation results show that the efficiency of the developed guidance law for the midcourse guidance is comparable with the performance of the kappa law.

Remark: The coefficients of the preceding guidance law were chosen based on the LOS angle rates estimates as indicated in Ref. 2. Their tuning can bring even better results than obtained, especially if we choose separately the guidance law coefficients for low elevation targets. For targets at close distances and low altitudes, the guidance law enabled us to obtain faster performance (time of intercept) than the kappa algorithm without loss of terminal velocity.

IV. Conclusions

The analytical expressions of the guidance law were obtained for the generalized planar and three-dimensional engagement models. It was shown that the three-dimensional Lyapunov-based guidance law embeds the Lyapunov-based guidance laws obtained for the planar case.

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